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AN IMPROVED IMPLICIT FINITE-DIFFERENCE SCHEME FOR BOUNDARY-LAYER FLOWS

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SUMMARY

The paper presents a modification to two well-known non-iterative implicit finite-difference schemes for confined and unconfined boundary-layer-type flows. The modification aims at improving the accuracy of these schemes by reducing the adverse effect of linearization, which is inherent in both of them. Using the present improved scheme, the same level of accuracy of the results could be obtained with large mesh sizes in the flow direction (coarse grid). The modification is done by adding a local iterative procedure at each computational step in the flow (marching) direction. As an example, to demonstrate the proposed modification, the simple case of developing forced convection in the entry region of concentric annuli has been considered. The results are presented, which prove the applicability of the proposed modification and show its effect on the obtained accuracy and on the required computer time.

KEY WORDS Iterative Finite-difference Confined Unconfined Boundary-layer flows

INTRODUCTION

In 1955, Rouleau and Osterle[1] presented a non-iterative implicit finite-difference scheme for solving the momentum and mass conservation equations which govern the two-dimensional boundary-layer-type flows. Using this scheme, they obtained numerical solutions for two unconfined boundary-layer flows, namely, the problem of a longitudinal flow over a flat plate with an arbitrary distribution of suction at the plate surface and the problem of mixing of a two-dimensional jet and a parallel-uniform stream when the jet discharges parallel and adjacent to a flat plate.

In 1961, Bodoia and Osterle[2] extended the aforementioned finite-difference scheme to handle cases with a variable pressure in the direction of flow. As a result of this extension, they obtained numerical solutions for the developing viscous flow, with a uniform entrance velocity profile, through a straight channel formed by two parallel flat plates when the two plates are stationary

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and when one of them is moving parallel to the other. Then Bodoia and Osterle[3] used this extended finite-difference scheme for solving numerically the boundary-layer-type equations which govern the developing free convection between two heated vertical parallel plates.

Since these publications,[1-3] the above-mentioned non-iterative implicit finite-difference schemes have been extensively used or developed to obtain numerical solutions for the boundary-layer-type equations which may describe the flow in many cases. Such cases include, for example, developing forced, mixed or induced flows in the entry region of ducts of various geometrical shapes under a variety of boundary conditions and flows about stationary or rotating bodies of revolution in fluid streams. Only some of the vast amount of published papers which have utilized numerical procedures based on the non-iterative implicit finite-difference schemes of Rouleau and Osterle[1] and Bodoia and Osterle[2,3] will be reviewed hereinafter.

Shohet[4,5] used the scheme of Bodoia and Osterle[2] and obtained numerical solutions for the magnetohydrodynamic entrance region problem in annular channels. Hornbeck *et al.*[6] solved the entry region problem in porous tubes. Magnetohydrodynamic and forced convection flows in the entrance region of flat rectangular ducts were investigated by Hwang and Fan.[7,8] Hornbeck[9-11] and Sutterby[12] treated the laminar flow problem at the entrance region of circular tubes and conical ducts, respectively. Coney and El-Shaarawi[13] presented results for the problem of incompressible laminar flow heat transfer in concentric annuli with simultaneously developing hydrodynamic and thermal boundary layers, the boundary conditions of one wall being isothermal and the other being adiabatic.

Lawrence and Chato[14] used the same scheme (of Bodoia and Osterle[2]) and obtained a numerical solution to the boundary-layer-type equations for the developing combined forced–free laminar flow in a vertical tube with a uniform velocity profile at the entrance and a constant wall heat flux. Marner and McMillan[15] treated the same problem, but with a constant wall temperature. Sherwin and Wallis[16,17] and El-Shaarawi and Sarhan[18] presented results for the problem of combined forced–free laminar convection in the entrance region of vertical concentric annuli.

The developing natural convection in vertical ducts has been treated by several investigators using the non-iterative finite-difference scheme of Bodoia and Osterle.[3] Dyer and Fowler,[19] Aung *et al.*,[20] Miyatake and Fujii[21,22] and Miyatake *et al.*[23] investigated the developing free convection between heated vertical parallel plates with various boundary and inlet conditions. Kageyama and Izumi,[24] Davis and Perona,[25] and Dyer[26] dealt with the developing free convection in vertical circular tubes both with constant heat flux and with constant temperature boundary conditions. Dyer[27] investigated the effect of an inlet restriction on the developing free convection flow in circular tubes. With one boundary being adiabatic while the opposite boundary is isothermal or at constant heat flux, the developing free convection in open-ended vertical concentric annuli was investigated by El-Shaarawi and Sarhan[28] and Al-Arabi *et al.*,[29] respectively.

Coney and El-Shaarawi[30] presented an indirect extension of the original work of Bodoia and Osterle[2] to include the case of rotating boundaries. Then, they[31] presented results for laminar flow heat transfer in the entrance region of concentric annuli with rotating inner walls. El-Shaarawi and Sarhan[32] used this extension and presented results for the problem of mixed convection in vertical annuli with rotating inner walls. El-Shaarawi and Sarhan[33] and El-Shaarawi and Khamis[34] considered, respectively, the developing laminar free convection in isothermally heated and uniformly heated vertical annuli with rotating inner walls.

El-Shaarawi *et al.*[35] presented an extension of the finite-difference scheme of Rouleau and Osterle[1] to deal with unconfined flows with rotating boundaries. Recently, El-Shaarawi *et al.*[36] applied this extended finite-difference scheme to the problem of mixed convection about a rotating sphere in an axial stream.

The main objective of this paper is to present a simple modification to the non-iterative implicit finite-difference schemes of Rouleau and Osterle[1] and Bodoia and Osterle.[2,3] Such a modification aims at increasing the accuracy of these schemes without decreasing the mesh sizes in the flow direction. This is done by including an iterative procedure at each step in the flow (boundary-layer) direction. The iterative process is executed, at each station and before moving to the next station in the marching (flow or boundary-layer) direction, by readjusting the values of the linearized coefficients in the inertia terms of the finite-difference equation(s) corresponding to the momentum equation(s). Such a readjustment is done iteratively between the values of the velocity components at the station under consideration and the corresponding values at the previous station. Thus, this proposed iterative procedure depends on the upstream flow values, has no relation with the corresponding downstream values, and may, hence, be applied at all or some selected stations in the flow direction. It can, therefore, be considered as a local or an internal iterative process within the numerical scheme. It will reduce the adverse effect of linearization inherent in the schemes of Rouleau and Osterle[1] and Bodoia and Osterle.[2]

The proposed modification can easily be applied to any of the previously mentioned extensions which deal with cases having rotating boundaries. However, for the purpose of demonstration, the simple case of a developing forced convection in the entry region of a stationary concentric annulus has been chosen. This case, which has previously been solved by several investigators,[13,37] will be utilized to explain the idea of the modification and discuss its effect on the accuracy of the results and on the required computer time. In order to achieve such a clarification, and also for the sake of completeness, the governing differential equations, their corresponding finite-difference representations, the method of the numerical solution using Bodoia and Osterle's scheme, the main features of this scheme and also of the scheme of Rouleau and Osterle, which stimulated the idea of modification, will first be briefly reviewed. Then, the proposed modification will be presented and its effect on the accuracy of the results and on the computation time will be displayed.

ON THE SCHEMES OF ROULEAU AND OSTERLE[1] AND BODOIA AND OSTERLE[2,3]

One of the main advantages of the implicit finite-difference scheme of Rouleau and Osterle[1] for unconfined flows is that the difference equations are linearized and locally uncoupled. This is achieved by assuming that where the product of two unknowns occurs, one of them is given approximately by its value at the previous position. Thus, the continuity equation becomes locally (i.e. within one step in the marching direction) uncoupled from the momentum equation and its solution may be deferred until the momentum equation is solved and the velocity component in the direction of flow is obtained. Then, the continuity equation can be solved to obtain the velocity component in the normal direction.

On the other hand, if the flow is confined, it may be difficult to determine the pressure distribution at the outer edge of the boundary layer and, hence, the pressure gradient in the flow direction within the boundary layer remains unknown in the boundary-layer flow model. Thus, in such a case, the simplified (boundary-layer) flow model includes only two equations in three unknowns (the two velocity components and the pressure). To avoid this obstacle, Bodoia and Osterle[2,3] used the integral continuity equation together with a linearized finite-difference form of the momentum equation as two equations in only two unknowns (the pressure and the velocity component in the flow direction). This is achieved by linearizing the inertia terms in the momentum equations. Similar to the case of unconfined flows, such a linearization is done by assuming that where the product of two unknowns occurs, one of them is replaced by its value at the previous position. Thus, the value of the normal velocity component, which appears in one of the inertia terms of the momentum equation, is considered known from the previous step. After computing the pressure and the velocity component in the direction of flow at each marching station, the actual values of the normal velocity component at that station are obtained from a difference form of the continuity differential equation.

As can be seen from the previous discussion, both the finite-difference schemes of Rouleau and Osterle[1] and Bodoia and Osterle[2,3] depend mainly on the linearization of the inertia terms of the momentum equation. Therefore, the accuracy of the obtained velocity components, and specially the normal velocity component, can always be increased by using small mesh sizes in the marching direction (in order to reduce the adverse effect of linearization). However, it may be worth mentioning that, for entrance region flows, the boundary-layer equations become asymptotically exact (i.e. identical to the original Navier–Stokes equations of motion) as the flow moves away from the entrance. This is because (1) as the flow moves away from the entrance, it approaches full development and, hence, the diffusion of momentum in the flow direction (which is neglected by boundary-layer assumptions) becomes vanishingly small, and (2) the inertia terms gradually vanish. Thus, as a result of this second reason, it is expected that the adverse effect of linearization will be reduced as the flow moves away from the entrance.

On the other hand, one would like to avoid or reduce the adverse effect of linearizing the inertia terms all over the domain of solution, particularly near the point at which the boundary-layer(s) starts its formation. The present modification has achieved this goal, as will be clarified by the results given at the end of this paper. This modification comprises an adjustment of the linearized coefficients through an iterative procedure at each step in the main flow (boundary-layer) direction instead of directly advancing to the next step as in the schemes of References 1–3.

A CASE FOR DEMONSTRATION

Consider steady, rotationally symmetric, laminar flow of an incompressible Newtonian fluid with constant physical properties in the entry region of a concentric annulus having an inner radius r_1 and an outer radius r_2. Figure 1 illustrates the geometry, co-ordinate system, and the finite-difference grid used. Let the inner wall of the annulus be isothermally heated to a temperature, t_{W_1} which is greater than that of the ambient, t_0, while the outer wall is perfectly insulated. Fluid is pumped into the annular gap between the two cylindrical walls and is assumed to enter the channel with a uniform velocity profile at a value equal to the mean axial velocity in the annular gap, u_0, and with a uniform temperature profile at a value equal to the ambient temperature, t_0. The flow is without internal heat generation and both viscous dissipation and axial conduction (diffusion) of heat are neglected. Further, applying Prandtls' boundary-layer assumptions, which are valid if the inertial forces are large relative to the viscous forces, the conservation equations for mass, momentum and energy which govern the case under consideration are, respectively, in their dimensionless forms, as follows:

$$\frac{\partial V}{\partial R}+\frac{V}{R}+\frac{\partial U}{\partial Z}=0, \tag{1}$$

$$\underline{V}\frac{\partial U}{\partial R}+\underline{U}\frac{\partial U}{\partial Z}=-\frac{\partial P}{\partial Z}+\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}, \tag{2}$$

$$\underline{V}\frac{\partial T}{\partial R}+\underline{U}\frac{\partial T}{\partial Z}=\frac{1}{Pr}\left(\frac{\partial^2 T}{\partial R^2}+\frac{1}{R}\frac{\partial T}{\partial R}\right). \tag{3}$$

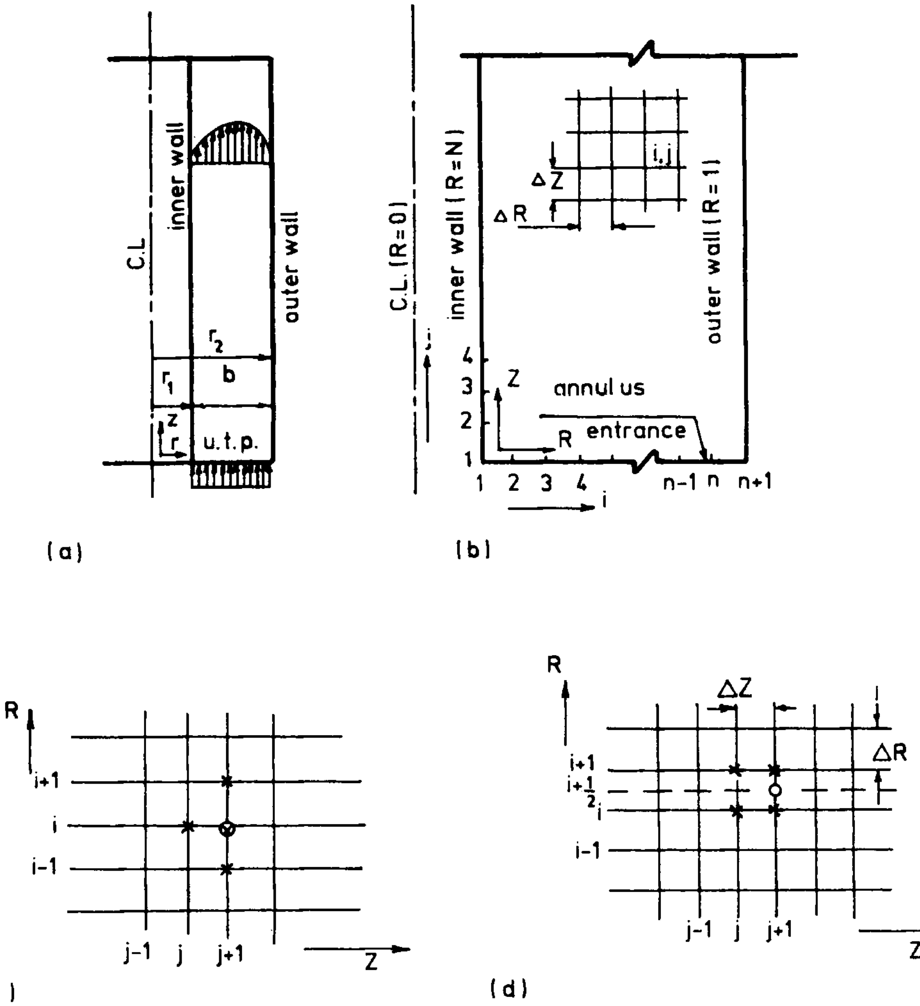

Figure 1. (a) Two-dimensional channel; (b) finite-difference network; (c) grid points for the momentum and energy equations; (d) grid points for the continuity equation

In the above equations $R=r/r_2$, $U=u/u_0$, $P=(p-p_0)/(\rho u_0^2)$, $V=\rho v r_2/\mu$, $Z=2z(1-N)/(r_2\, Re)$, $T=(t-t_0)/(t_W-t_0)$ and $Pr=\mu c/k$, where c is the specific heat of fluid under constant pressure, k is its thermal conductivity, N is the annulus radius ratio $(=r_1/r_2)$, p is the fluid pressure, r is the radial co-ordinate, Re is the Reynolds number $[=2(r_2-r_1)\,\rho u_0/\mu]$, t is the fluid temperature, u is the axial velocity component, v is the radial velocity component, z is the axial co-ordinate, μ is the fluid viscosity, ρ is the fluid density, and the subscript o denotes 'at entrance cross-section'. The assumption of constant physical properties uncouples the energy equation (3) from the continuity and momentum equations, (1) and (2), respectively.

In Figures 1(c) and 1(d), parts of the finite-difference domain are drawn to show the points involved in transforming each differential equation into its finite-difference form. Thus, the finite-difference representations of the above equations are, respectively, as follows:

$$\frac{V_{i+1,j+1}-V_{i,j+1}}{\Delta R}+\frac{V_{i+1,j+1}+V_{i,j+1}}{2[N+(i-1/2)\Delta R]}+\frac{U_{i+1,j+1}+U_{i,j+1}-U_{i+1,j}-U_{i,j}}{2\Delta Z}=0, \tag{4}$$

$$\underline{V_{i,j}}\frac{U_{i+1,j+1}-U_{i-1,j+1}}{2\Delta R}+\underline{U_{i,j}}\frac{U_{i,j+1}-U_{i,j}}{\Delta Z}$$

$$=\frac{P_j-P_{j+1}}{\Delta Z}+\frac{U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}}{(\Delta R)^2}+\frac{1}{(N+i\Delta R)}\frac{U_{i+1,j+1}-U_{i-1,j+1}}{2\Delta R}, \tag{5}$$

$$\underline{V_{i,j}}\frac{T_{i+1,j+1}-T_{i-1,j+1}}{2\Delta R}+\underline{U_{i,j}}\frac{T_{i,j+1}-T_{i,j}}{\Delta Z}$$

$$=\frac{1}{Pr}\frac{T_{i+1,j+1}-2T_{i,j+1}+T_{i-1,j+1}}{(\Delta R)^2}+\frac{T_{i+1,j-1}-T_{i-1,j+1}}{2\Delta R[N+(i-1)\Delta R]}. \tag{6}$$

The above equations are subject to the following boundary conditions:

$$U_{i,j}=U_{n+1,j}=V_{1,j}=V_{n+1,j}=T_{i,1}=P_1=0,.$$
$$T_{1,j}=1 \quad \text{and} \quad T_{n,j}=T_{n+2,j}.$$

The linearization of the first inertia term in equation (5) makes this equation as if it were with only two unknowns (U and P), since the value of V is taken as known from the previous axial step. Thus, if we have another governing equation which contains U only, then such an equation with equation (5) will form a complete system of two equations in two unknowns. Indeed, the following integral continuity equation satisfies this requirement:

$$\int_N^1 UR\,\mathrm{d}R=(1-N^2)/2. \tag{7}$$

Rewriting this equation using the trapezoidal rule and applying the boundary conditions $U_{i,j}=U_{n+1,j}=0$, one obtains

$$\Delta R\sum_{i=2}^{n} U_{i,j}[N+(i-1)R]=(1-N^2)/2. \tag{8}$$

Now, equations (5) and (8) can be solved together to find the unknown values of U's and P (with subscripts $j+1$). Having obtained the values of U, the correct values of V at the cross-section under consideration can be computed by means of equation (4). Repeating this process, in the scheme of Bodoia and Osterle, we can advance step by step downstream.

It is instructive to mention that equation (7) is in fact an integrated form of equation (1) subject to the appropriate boundary conditions. Therefore, equations (1) and (7) are not independent relationships and it may appear that we are computing three variables (U, V and P) by means of only two equations (the simplified z-momentum equation and the continuity equation). However, this is not true since the cornerstone of Bodoia and Osterle's linearized finite-difference technique is the application of equations (5) and (8) at each cross-section with the values of V in equation (5) as taken from the previous axial step, i.e. equations (5) and (8) are considered as two equations in only two unknowns (U and P). Then, after computing U and P at each cross-section, the values of V at that cross-section are obtained by means of equation (4) with the known values of U. Therefore, it is clear that the accuracy of the obtained values of V can always be increased by using small axial mesh sizes (particularly near the entrance, where the gradients and values of V are expected to be large), which at the same time implies a reduction in the adverse effect of the linearization with respect to U in the finite-difference representation (5) of equation (2). However, such a goal may be achieved without using small axial mesh sizes through utilization of the following modified scheme.

Figure 2 gives a flow chart illustrating the proposed modified scheme. The numerical procedure for the case under consideration is summarized as follows. Starting with $j=1$ (entrance cross-section) and applying equation (5), with $i=2, 3, \ldots, n$, and equation (8) to the entire cross-section we get n simultaneous linear algebraic equations which when solved give the values of n unknowns, namely, $U_{2,2}, U_{3,2}, \ldots, U_{n-1,2}, U_{n,2}$, and P_2. Using these computed values of U's and applying equation (4), we get the values of V's at all points of the second column. At this stage the obtained values of U's and V's will be used to adjust the linearized (underlined) coefficients of the inertia terms of equation (5) in order to obtain more accurate values of U's, P and, hence, V's at the second cross-section. Such a return (with the new values of U and V) to the momentum equation will be repeated until the computed values of U, V, and P at the second cross-section do not practically change with iteration. In other words, the iteration process continues until the change in the value of any of the computed unknowns within the consequent iterations does not exceed a prespecified value. When such a condition is achieved and the iteration across the axial step is completed, we can solve the energy equation (6) with the underlined coefficients adjusted using the obtained values of $U_{i,2}$ and $V_{i,2}$ (instead of the linearized values $U_{i,1}$ and $V_{i,1}$). Then,

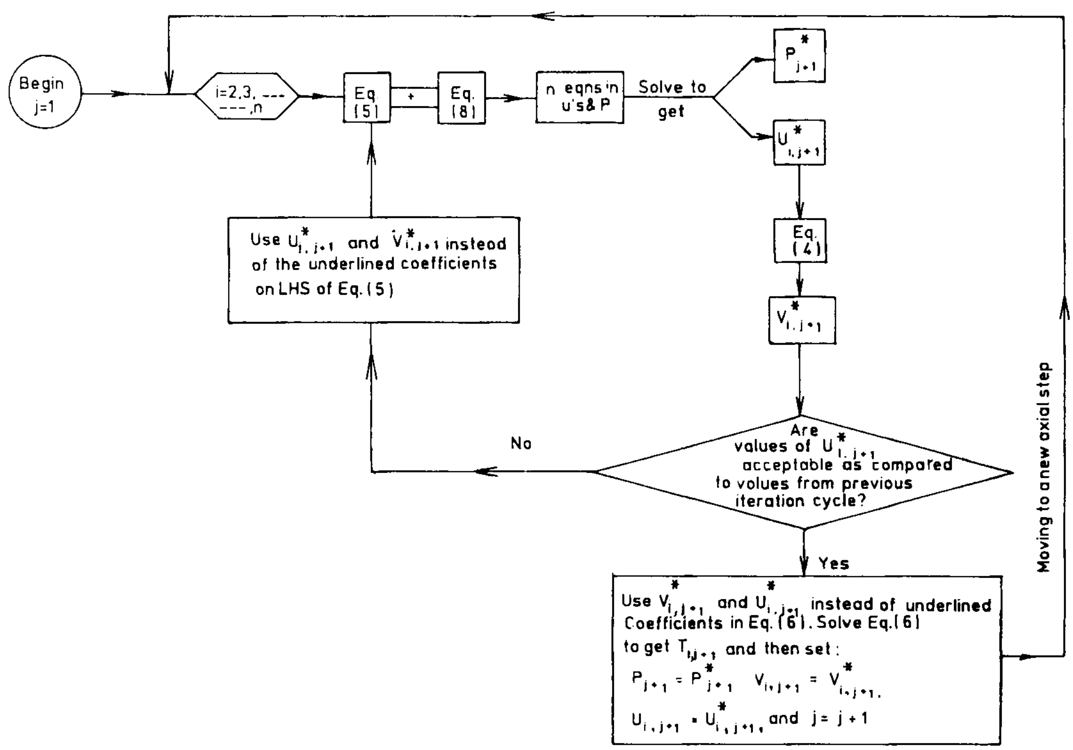

Figure 2. Flow chart for the modified scheme

we can advance one step in the marching (axial) direction and repeat the whole cycle with $j=2$. Thus, the computations proceed step by step until full development is achieved.

To differentiate between the final acceptable values of an unknown, which are obtained at the end of the iteration process, and its (intermediate) values during the iteration process, the latter have been superscripted by an asterisk in the flow chart of Figure 2. Thus, the computations are at every new axial step using equation (5). Then, after obtaining the temporary (iterative) values of U and P, equation (5) is replaced (during the iterative process) by the following equation:

$$V^*_{i,j+1}\,\frac{U_{i+1,j+1}-U_{i-1,j+1}}{2\Delta R}+U^*_{i,j+1}\,\frac{U_{i,j+1}-U_{i,j}}{\Delta Z}=\text{RHS of equation (5).} \tag{9}$$

On the other hand, it is possible, in the present forced convection case [since there is no coupling between the energy equation (3), i.e. equation (6), and the momentum equation (2), i.e. equation (5)], to use the following equation directly instead of equation (6):

$$V_{i,j+1}\,\frac{T_{i+1,j+1}-T_{i-1,j+1}}{2\Delta R}+U_{i,j+1}\,\frac{T_{i,j+1}-T_{i,j}}{\Delta Z}=\text{RHS of equation (6).} \tag{10}$$

However, it should be emphasized that in cases of mixed or free convection, the energy equation will be coupled with the momentum equation and it will not be possible to use equation (10). In such cases, the computation should start for each new axial step using equation (6). Then, after obtaining the first temporary (iterative) values of U, P, and V, equation (6) is replaced (during the iterative process) by the following equation:

$$V^*_{i,j+1}\,\frac{T_{i+1,j-1}-T_{i-1,j+1}}{2\Delta R}+U^*_{i,j+1}\,\frac{T_{i,j+1}-T_{i,j}}{\Delta Z}=\text{RHS of equation (6).} \tag{11}$$

Thus, the principle of the present modified scheme is to employ two difference equations for each of the momentum and energy equations.

RESULTS AND DISCUSSION

It should be noted that the present results do not aim at giving fluid-flow and heat-transfer data for the case under consideration; such data are available in the literature as this case has already been solved by several investigators.[13,37] In fact, the aim of the present results is to demonstrate the applicability of the proposed modified finite-difference scheme and to display the effect of such a modification on the accuracy of Bodoia and Osterle's scheme and on the required computer time. In order to achieve such a comparative task, it was decided to fix the number of radial increments equal to 20 over the entire development region in all the computer runs. Moreover, in all the computer runs which employ the modified scheme, the iteration process ends if the computed values of the axial velocity component ($U^*_{i,j+1}$) show little further change with iteration; a maximum tolerable value of 0·005 per cent was chosen as the percentage change in $U^*_{i,j+1}$ between any two successive iterations.

All the computations were carried out in an annulus of radius ratio $N=0{\cdot}9$ for a fluid of $Pr=0{\cdot}7$. The first computer run was made employing the modified scheme (which includes iteration across each axial step) with an axial mesh size $\Delta Z=10^{-6}$ near the entrance and then the axial step was increased several times as the flow moves away from the entrance (approaching full development). A second computer run was made employing Bodoia and Osterle's scheme (without iteration) and using exactly the same mesh sizes of the first computer run (so that the order of truncation-error magnitude would be the same as in the first run). Each of these two runs was then repeated three times but with axial mesh sizes equal to, respectively, five, ten, and 20 times the axial mesh sizes of the first computer run. These additional six runs were carried out in order to investigate the effect of the axial mesh size on the accuracy of the results and on the computer time needed in both cases (with and without iteration). The results obtained from the first computer run are known to be the most accurate among the series of eight computer runs since this run has the smallest mesh sizes and also includes the iteration process across each axial step. Therefore, the results of the first run will be considered a base to which all other results will be referred. So, denoting the axial mesh sizes of the first run by ΔZ_1, the second, third, and fourth runs, in either the group of runs with iteration or in the group without iteration, will have values of $\Delta Z^* = \Delta Z/\Delta Z_1 = 5$, 10, and 20, respectively.

Engineers are not frequently concerned with the details of the fluid velocity and temperature profiles but only with the pressure drop and the mixing cup (mixed-mean) fluid temperature. The latter is defined by the equation

$$t_m = \int_{r_1}^{r_2} rut \, dr \bigg/ \int_{r_1}^{r_2} ur \, dr, \tag{12}$$

which in dimensionless form is

$$T_m = \int_N^1 RUT \, dR \bigg/ \int_N^1 RU \, dR. \tag{13}$$

Such a concern exists because the pressure drop determines the required pumping power while the mixing cup temperature can be used to evaluate the heat transfer to the fluid between the annulus entrance and any other cross-section downstream. Therefore, the present work has focused on these two quantities.

Tables I and II give, respectively, the variations of the dimensionless pressure drop, $-P$, and the mixing cup temperature, T_m, with the dimensionless axial distance, Z, for the previously mentioned eight computer runs. Table III refers the results of these two quantities, in the region

Table I. Variation of pressure ($-P$) with Z for eight computer runs

	Modified scheme (with iteration)				Bodoia and Osterle's scheme			
$z \times 10^5$ \ Δz^*	1	5	10	20	1	5	10	20
2	0·19899	0·19502	0·19098	0·18434	0·20558	0·21338	0·17807	0·11417
6	0·31030	0·30992	0·30830	0·30454	0·31421	0·32706	0·33474	0·32980
10	0·39281	0·39333	0·39283	0·39126	0·39582	0·40818	0·41722	0·42543
30	0·69994	0·70101	0·70073	0·70054	0·70227	0·71223	0·72162	0·73388
50	0·95560	0·95680	0·95745	0·95848	0·95781	0·96737	0·97698	0·98969
90	1·44103	1·44247	1·44348	1·44512	1·44328	1·45281	1·46329	1·47710
150	2·16296	2·16445	2·16573	2·16779	2·16521	2·17479	2·18550	2·19975
250	3·36575	3·36725	3·36854	3·37064	3·36800	3·37758	3·38831	3·40263
350	4·56854	4·57004	4·57133	4·57344	4·57079	4·58037	4·59110	4·60543
450	5·77133	5·77282	5·77412	5·77623	5·77358	5·78315	5·79389	5·80822
650	8·17690	8·17840	8·17969	8·18180	8·17915	8·18873	8·19946	8·21379
1150	14·19084	14·19233	14·19363	14·19574	14·19309	14·20266	14·21340	14·22773
1650	20·20478	20·20580	20·20757	20·20967	20·20702	20·21030	20·22733	20·24166
3150	38·24659	38·24770	38·24938	38·25148	38·24883	38·25740	38·26914	38·28347
8150	98·38595	98·38710	98·38874	98·39085	98·38820	98·39100	98·40851	98·42284

Table II. Variation of T_m with Z for eight computer runs.

	Modified scheme (with iteration)				Bodoia and Osterle's scheme			
$z \times 10^5$ \ Δz^*	1	5	10	20	1	5	10	20
2	0·03590	0·03636	0·03582	0·03407	0·03569	0·03315	0·03527	0·02752
6	0·06878	0·06990	0·07069	0·07104	0·06886	0·06940	0·07050	0·06639
10	0·09202	0·09319	0·09402	0·09476	0·09211	0·09322	0·09406	0·09092
30	0·17395	0·17478	0·17519	0·17561	0·17404	0·17521	0·17521	0·17953
50	0·23614	0·23674	0·23710	0·23735	0·23621	0·23723	0·23714	0·24108
90	0·33766	0·33792	0·33688	0·33547	0·33773	0·33835	0·33694	0·33885
150	0·45988	0·45939	0·45787	0·45520	0·45993	0·45973	0·45793	0·45788
250	0·61290	0·61258	0·60268	0·59239	0·61293	0·61283	0·60273	0·59436
450	0·80093	0·79813	0·78629	0·77148	0·80095	0·79826	0·78631	0·77257
650	0·89763	0·89397	0·88503	0·87183	0·89763	0·89404	0·88504	0·87245
1150	0·97953	0·97880	0·96597	0·95211	0·97827	0·97882	0·96598	0·95234
1650	0·99591	0·99532	0·98993	0·98211	0·99787	0·99532	0·98993	0·98219
3150	0·99994	0·99984	0·99894	0·99907	0·99996	0·99984	0·99894	0·99907

Table III. Normalized pressure drop against axial distance very near to the entrance

	$P/P_1 \times 100\%$						
	Modified scheme			Bodoia and Osterle's scheme			
$z \times 10^5$ \ Δz^*	5	10	20	1	5	10	20
2	98·00	95·97	92·63	103·31	107·23	89·49	57·37
4	99·36	98·34	96·06	101·86	106·51	108·18	92·33
6	99·88	99·36	98·14	101·26	105·40	107·88	106·28
8	100·06	99·82	99·11	100·95	104·54	107·02	108·72
10	100·13	100·01	99·61	100·77	103·91	106·21	108·30
18	100·20	100·03	99·82	100·48	102·58	104·32	106·47
30	100·15	100·11	100·09	100·33	101·76	103·10	104·85
			$T_m/T_{m_1} \times 100\%$				
2	101·28	99·78	94·90	99·42	92·34	98·25	76·66
6	101·76	102·78	103·29	100·12	100·90	102·50	96·53
10	101·27	102·17	102·98	100·10	101·30	102·22	98·80
18	100·80	101·22	101·67	100·09	101·04	101·26	102·84
30	100·48	100·71	100·95	100·05	100·72	100·72	103·21
50	100·25	100·41	100·51	100·03	100·46	100·42	102·09

near the annulus entrance, to their corresponding values which were obtained in the first computer run (with the smallest mesh sizes and using the modified scheme). Thus, in Table III, the results of the first computer run represent the 100 per cent base to which the corresponding results of all the other computer runs are referred. As can be seen from these tables, for either group of computer runs (the group using the modified scheme or that using Bodoia and Osterle's scheme), the accuracy of the obtained results generally increases as the axial mesh sizes decrease. This is as might be expected and is attributed to the decrease in the discretization error as the mesh sizes

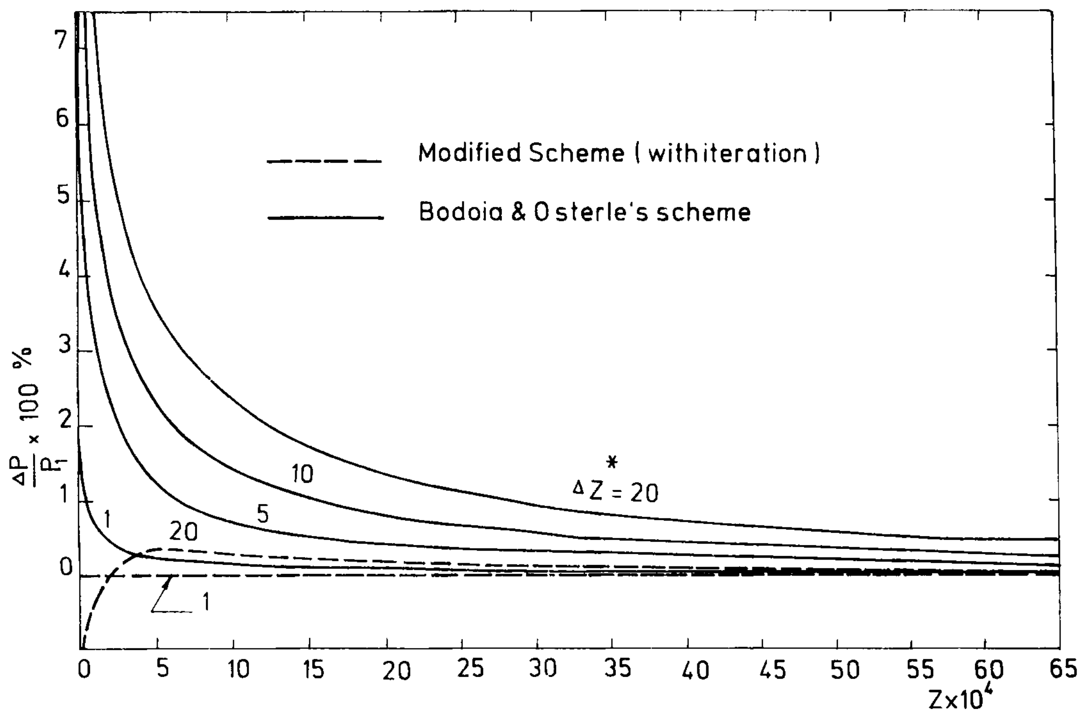

Figure 3. Modified scheme with iteration (– – –); Bodoia and Osterle's scheme (——)

decrease. Moreover, the percentage deviation in the results of any computer run as referred to the first computer-run results [i.e. $\Delta P/P_1 \times 100\% = (P - P_1)/P_1 \times 100\%$ or $\Delta T_m/T_m \times 100\%$] shows, as depicted in Figure 3 [apart from the slight unpredicative vibrations very near to the entrance (presented in Table III)], smooth decay with an increase in the value of Z (i.e. as the flow approaches full development). This behaviour may be attributed to the decrease in the adverse effect of linearization as the flow approaches the state of full development.

Comparing the results of the group of computer runs which employ the modified scheme with those of the group which uses Bodoia and Osterle's scheme, the following remarks can be made. The accuracy of Bodoia and Osterle's scheme (scheme without iteration) decreases quite rapidly as the axial mesh size increases, especially in regions very close to the entrance. It is interesting to note that with $\Delta Z^* = 20$, the result obtained for the pressure drop at $Z = 2 \times 10^{-5}$ from the computer run which employs Bodoia and Osterle's scheme is only 55·5 per cent of its corresponding value with $\Delta Z^* = 1$ and about 57 per cent of the corresponding value obtained from the computer run employing the modified scheme with $\Delta Z^* = 1$. However, employing the modified scheme with the same $\Delta Z^* = 20$, the corresponding result for the pressure drop at $Z = 2 \times 10^{-5}$ is improved to 92·6 per cent of the reference pressure drop (with $\Delta Z^* = 1$). Similar conclusions concerning the accuracy of the obtained mixing cup temperature can be deduced from Tables II and III. The rapid decrease in the accuracy of the obtained results with large values of ΔZ^* in case of Bodoia and Osterle's scheme as compared with the modified scheme can be attributed to the adverse effect of linearizing the terms on the LHS of the governing equation; the modified scheme has compensated for this adverse effect by the iteration process across each axial step. On the other hand, it can also be seen from the results presented in the tables that with values of

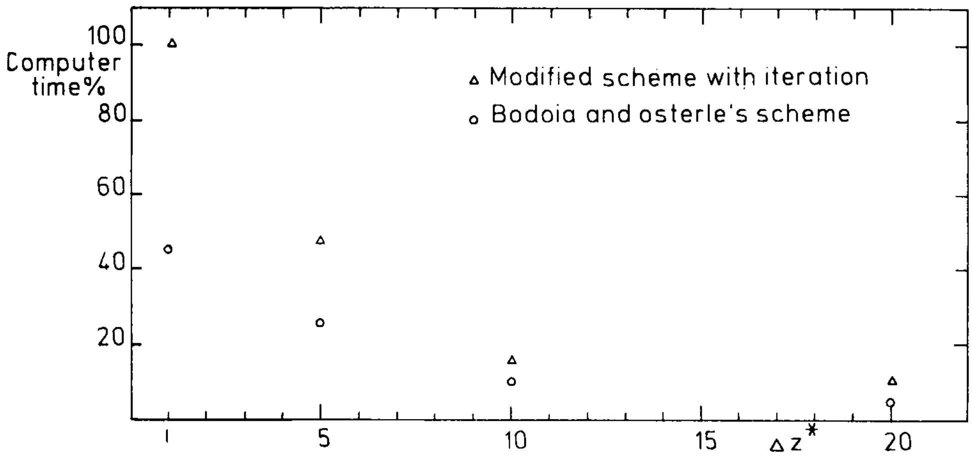

Figure 4. Modified scheme with iteration (△); Bodoia and Osterle's scheme (○)

$Z > 8 \times 10^{-5}$, the pressure drop results by the modified scheme remain within less than one per cent deviation even for mesh sizes 20 times larger. However, it is important to mention that with $\Delta Z^* = 1$, the results obtained by means of Bodoia and Osterle's scheme are in good agreement with those obtained using the modified scheme; the maximum deviations in the pressure drop and the mixing cup temperature are only 3·31 and 0·58 per cent, respectively. This means that the modified scheme is only useful when large mesh sizes in the flow direction are used.

It is important to compare the computer times required in all the previous computer runs so that the economic aspect can be taken into consideration when evaluation is needed. Figure 4 gives these computer times against ΔZ^*; all times are referred to that of the first computer run (the base run with the smallest mesh sizes and using the modified scheme). The computer time of any of the runs was taken as that time required for the mixing cup temperature to reach 99·995 per cent of its fully developed value (unity). As can be seen from this figure, the difference between the consumed computation times in the two cases (modified scheme and non-modified scheme) decreases as the axial mesh size increases. Taking into consideration that with large mesh sizes the accuracy of the modified scheme is much better than that of Bodoia and Osterle's scheme, the last point regarding the computation time justifies again the use of the modified scheme when large mesh sizes in the flow direction are employed. It may be important to mention that the results presented in Figure 4 show that with $\Delta Z^* = 10$, the time consumed by a computer run employing the modified scheme is less than 35 per cent that required by a computer run using Bodoia and Osterle's scheme with $\Delta Z^* = 1$, even though the accuracy in both cases is comparable.

Finally, it should be stated again that the case considered in this paper was only an example to demonstrate the modified scheme and its features. It is anticipated that the conclusions drawn from such an example are applicable to all other similar modifications applied to cases of unconfined flows or cases with rotating boundaries.

CONCLUSIONS

The paper presents a modification to the well-known non-iterative implicit finite-difference schemes of Bodoia and Osterle[2,3] and Rouleau and Osterle[1] for confined and unconfined boundary-layer flows, respectively. The modification depends on the application of an iterative process at each step in the marching (flow) direction. The simple case of a developing forced convection in the entry region of concentric annuli has been considered to demonstrate the applicability of the proposed modification and its effect on both the obtained accuracy and the required computer time. It has been found that the proposed modification is very useful, particularly near the duct entrance; it increases the accuracy of the obtained results considerably. For the same required computer time, it is possible to obtain results using the modified scheme as accurate as the corresponding results by the other schemes even though mesh sizes in the flow direction are several times larger in the former than in the latter case. For entrance region flows, it is recommended to use the modified scheme near the entrance and then use the non-iterative schemes near the fully developed region without considerably affecting the accuracy of the obtained results.

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